Final Exam - Optimization
B. Math III

27 April, 2026

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. In a classification problem, the statistician is provided with a sample of correctly classified data points:
$$\{(\vec{x}_1, y_1), \ldots, (\vec{x}_n, y_n)\},$$
where for each data point i, $\vec{x}_i \in \mathbb{R}^d$ is the vector of features and $y_i \in \{\pm 1\}$ is the class label. A *discriminant* function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies
$$\begin{cases} f(\vec{x}_i) > 0 & \text{if } y_i = +1 \\ f(\vec{x}_i) < 0 & \text{if } y_i = -1 \end{cases}$$
Such a function f can be used to classify new data: given a new feature vector $\vec{x} \in \mathbb{R}^d$, if $f(\vec{x}) > 0$, assign $\vec{x}$ to class $+1$, and if $f(\vec{x}) < 0$, assign $\vec{x}$ to class -1. Affine discriminant functions of this form are known as **Support Vector Machines**.
   (a) (10 points) Let $I := \{i : y_i = +1\}$ and $J := \{j : y_j = -1\}$ be the sets of indices for each class. Show that there exists an affine discriminant function for the dataset if and only if the convex hulls of $\{\vec{x}_i : i \in I\}$ and $\{\vec{x}_j : j \in J\}$ are disjoint.

(b) (5 points) For a non-zero vector $\vec{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, show that the distance between the two hyperplanes $\{\vec{x} \in \mathbb{R}^d : \langle \vec{a}, \vec{x} \rangle + b = 1\}$ and $\{\vec{x} \in \mathbb{R}^d : \langle \vec{a}, \vec{x} \rangle + b = -1\}$ is given by $\frac{2}{\|\vec{a}\|_2}$.

Total for Question 1: 15

2. For each of the following functions, determine whether it is convex, concave or neither (with proper justification):
   (a) (3 points) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{>0}$;
   (b) (3 points) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{>0}$;
   (c) (4 points) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{>0}$.

Total for Question 2: 10

3. Consider a linear programming problem in standard form:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ is the constraint matrix and A_k denotes its k-th column. Suppose in a simplex iteration we have a basic feasible solution x associated with a basis B. Let x_k be a nonbasic variable with reduced cost $\bar{c}_k < 0$ and $B^{-1} A_k \leq 0$.
   (a) (5 points) Construct a direction $d \in \mathbb{R}^n$ such that $c^T d < 0$.
   (b) (5 points) Show that the ray $\{x + \lambda d \mid \lambda \geq 0\}$ is feasible for all $\lambda \geq 0$.
   (c) (5 points) Using the results from the previous parts, conclude that the optimal cost of the LP is $-\infty$.

Total for Question 3: 15

4. Consider the following partially completed simplex tableau for a **minimization** problem in standard form. The entries marked with letters are unknown parameters.

	x_1	x_2	x_3	x_4	x_5	x_6
$-d$	0	a	0	b	0	c
4	1	-1	0	2	0	e
2	0	f	1	g	0	-2
3	0	3	0	-1	1	h

State the most general conditions on the unknowns a, b, c, d, e, f, g, h such that:
   (a) (4 points) The current basis is feasible.
   (b) (4 points) The current basic feasible solution is optimal.
   (c) (4 points) The optimal cost is $-\infty$ (the problem is unbounded).
   (d) (4 points) The current basic feasible solution is degenerate.

(e) (4 points) There exists an alternative optimal solution (assuming the current basis is optimal).

Total for Question 4: 20

5. A furniture shop produces two types of items: **Tables** (x_1) and **Chairs** (x_2). The profit for each table is ₹300 and for each chair is ₹200. The production is limited by two resources:

- **Labor:** Each table requires 2 hours, and each chair requires 1 hour. Total available labor is 100 hours.
- **Wood:** Each table requires 1 unit, and each chair requires 1 unit. Total available wood is 80 units.

(a) (10 points) Suppose the optimal production plan is $x_1^* = 20$ and $x_2^* = 60$. Using **Complementary Slackness**, determine the optimal shadow prices or dual variables (y_1^*, y_2^*).

(b) (10 points) Should the shop owner pay ₹150 for an additional unit of wood? Should they pay ₹80 for an additional hour of labor? Justify your answers.

Total for Question 5: 20

6. (10 points) Check the optimality of the proposed solution using complementary slackness:

$$\text{maximize } 4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$$

subject to

$$\begin{aligned} x_1 - 4x_3 + 3x_4 + x_5 + x_6 &\le 10 \\ 5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 &\le 4 \\ 4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 &\le 4 \\ -x_2 + 2x_4 + x_5 - 5x_6 &\le 6 \\ -2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 &\le 12 \\ 2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 &\le 16 \\ x_1, \ldots, x_6 &\ge 0 \end{aligned}$$

Proposed solution: $x_1 = 0, x_2 = 0, x_3 = \frac{2}{3}, x_4 = \frac{5}{2}, x_5 = \frac{7}{2}, x_6 = \frac{1}{2}$.

Total for Question 6: 10

7. (20 points) Consider the convex optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

where $f_0, \ldots, f_m$ are convex functions and $\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i$ represents the problem domain. Suppose that Slater's condition holds for this problem; that is, there exists a point $x \in \operatorname{relint} \mathcal{D}$ such that $f_i(x) < 0$ for all $i = 1, \ldots, m$ and $Ax = b$. Prove that if the primal optimal value p^* is finite, then strong duality holds ($d^* = p^*$) and there exists a dual optimal solution (λ^*, ν^*).

Total for Question 7: 20